AC Theory — Gapped Notes

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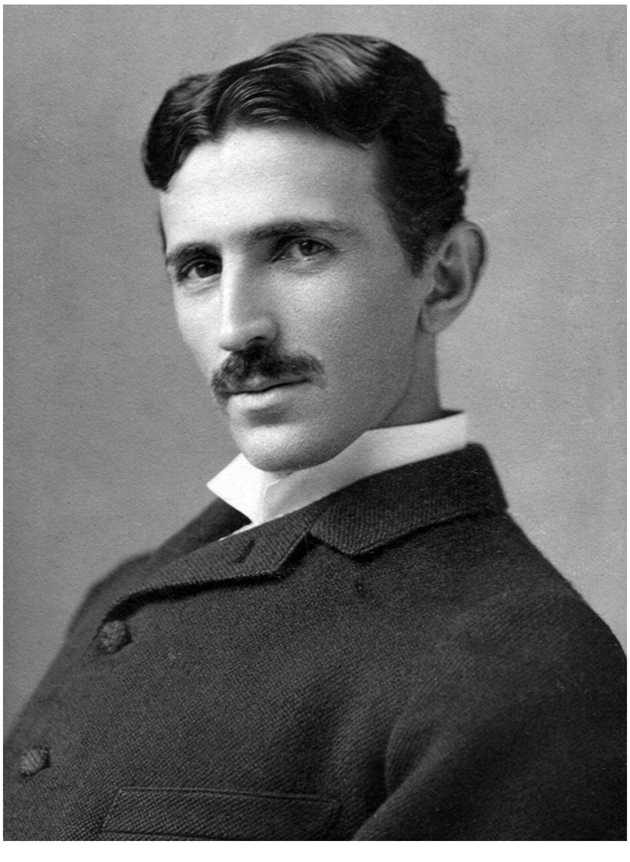


Nikola Tesla c.1890

What one man calls God, another calls the laws of physics. — Nikola Tesla

Contents

1 Introduction

These notes were written in conjunction with a lecture, delivered at the International Physics Olympiad (IPhO) team selection camp for the United Kingdom.

The subject introduced here is AC theory. The objective of these notes is to introduce various ideas of AC circuits, with sufficient mathematical rigour, yet without clouding the students' mind.

The key idea within this text, and beginning AC theory in general is that all the definitions, and everything being manipulated is purely for the sake of making calculations easier and generally more computable. Later in higher education, you will find that Laplace transforms turn all of this into trivial polynomial algebra. But you must traverse the whole journey up to that, to build intuition: Rome was not built in a day.

Finally, if something does not seem to make sense in the way it is explained in the notes, try reading other explanatory materials. There is no "best" point of view, and another may work better for you. Additional materials are referenced at the end of the notes.

1.1 Pre-requisites

The following are pre-requisites for self-studying from the notes. These should be studied at least to A-level standard. If in tandem with the lecture, all mathematical points will be explained in thorough detail.

- Ordinary linear differential equations
- Basic complex numbers
- Kirchhoff's current and voltage laws
- Differential equations approach to capacitors and inductors

Please contact ckz20@cam.ac.uk if you find any mistakes.

2 Notation

Notation is of great importance, for you to not only convey your ideas effectively to others, but also to yourself. In electrical circuit analysis, there are several conventions to be followed.

1. Uppercase letters will be used for DC.
2. Lowercase letters will be used for AC.

For example, V would indicate a DC voltage, whereas v would suggest an AC voltage (possibly a sinusoid). Following this convention, it is quick to see that AC current is given by i. This is a problem! We have always denoted the imaginary unit as i. To account for this, we will employ j. You will find this commonly within engineering. Phasors (to be described later) will be denoted with a tilde on top, i.e.

Root Mean Square (RMS) values will be denoted as uppercase letters, i.e.

This is due to their relevance to DC. You may have heard that a DC circuit with RMS values will deliver the same power as the original AC circuit. We will prove this later. Peak values (amplitudes) will have a hat, i.e.

3 Introducing Complex Numbers

We know that linear circuit components (resistors, capacitors and inductors) can be modelled with differential equations, i.e. for a series RLC circuit

However, these differential equations can be quite hard to solve. We need something a bit more straightforward. Let us define,

$$z = x + \mathrm{j}y \in \mathbb{C}, \text{ where } x, y \in \mathbb{R}.$$

For circuits that you will encounter in IPhO and generally pre-university, they will all be linear, with linear components, such that you can apply principles of superposition. To denote a linear system, we construct a linear differential equation in a more general form, where α_k are just constant real coefficients (α_k can also be complex but this makes it more conceptually challenging so is omitted). F and z are complex functions (functions that take complex values). F is known as the "Forcing Function" and z the "Response Function". This equation represents a linear system because the derivative is what is known as a linear operator, simply down to the following reason,

where f, g are functions and α, β are scalars. Now, returning to the general form of the linear differential equation, we can take the real parts of both,

$$\mathfrak{Re}\{\sum \alpha_k \cdot \frac{\mathrm{d}^k z}{\mathrm{d}t^k}\} = \mathfrak{Re}\{F(t)\},$$

$$\sum \mathfrak{Re}\{\alpha_k \cdot \frac{\mathrm{d}^k z}{\mathrm{d}t^k}\} = \mathfrak{Re}\{F(t)\}.$$

Since α_k is real,

$$\sum \alpha_k \cdot \mathfrak{Re}\{\frac{\mathrm{d}^k z}{\mathrm{d}t^k}\} = \mathfrak{Re}\{F(t)\},$$

$$\sum \alpha_k \cdot \mathfrak{Re}\{\frac{\mathrm{d}^k x}{\mathrm{d}t^k} + \mathrm{j}\frac{\mathrm{d}^k y}{\mathrm{d}t^k}\} = \mathfrak{Re}\{F(t)\},$$

$$\sum \alpha_k \cdot \frac{\mathrm{d}^k x}{\mathrm{d}t^k} = \mathfrak{Re}\{F(t)\},$$

$$\sum \alpha_k \cdot \frac{\mathrm{d}^k \mathfrak{Re}\{z\}}{\mathrm{d}t^k} = \mathfrak{Re}\{F(t)\}.$$

What does this tell us? This suggests that for any complex forcing-response function pairs, the real part effects are independent of the imaginary part effects. Or in other words, only the real part of the forcing function affects the real part of the response function.

Hence, let us define the real part to be what we measure in real life. This means, when doing the mathematical model, we can choose the imaginary parts to be whatever is convenient since it is arbitrary what values/functions they take. In some cases, this could mean the imaginary part is chosen to be zero. However, in AC analysis, this is rarely the easiest way. So, for example, let v_{real} be the real measured voltage signal, such that

We may define a complex function v, where $v_{\text{real}} = \mathfrak{Re}\{v\}$, and carry out any calculations for v as a complex number. To acquire the real result we take real parts at the end. As said above, we can choose the imaginary part at our pleasing. For sinusoidal AC circuits, it is generally convenient to use Euler's Equation, since exponential functions tend to be easier to work with than trigonometric functions. Euler's Formula states,

so for the aforementioned v_{real}, we can have,

3.1 Example: Simple Capacitor Circuit

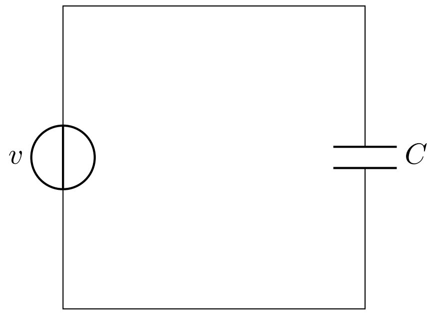


Figure 1: A simple capacitor circuit.

To provide intuition, let us do a simple calculation using this idea. Note, we have assumed the sign of the potential drop across the capacitor. Please see Cullerne and Machacek [2] section 5.3.5 should clarification be necessary (it will however be touched on in this set of notes later). We shall analyse the circuit in Figure 1. Firstly, analysis with no complex numbers, where $v = \hat{V}\cos\omega t$.

Now let us repeat this using complex numbers, where $v = \hat{V}e^{j\omega t}$,

Both methods lead to the same result.

3.2 Aside: Proof of Euler's Formula

Let $f(\theta) = e^{-\mathrm{j}\theta}(\cos\theta + \mathrm{j}\sin\theta)$,

$$\frac{\mathrm{d}f}{\mathrm{d}\theta} = -\mathrm{j}e^{-\mathrm{j}\theta}(\cos\theta + \mathrm{j}\sin\theta) + e^{-\mathrm{j}\theta}(-\sin\theta + \mathrm{j}\cos\theta),$$

$$\frac{\mathrm{d}f}{\mathrm{d}\theta} = e^{-\mathrm{j}\theta}(-\mathrm{j}\cos\theta + \sin\theta) + e^{-\mathrm{j}\theta}(-\sin\theta + \mathrm{j}\cos\theta) = 0.$$

This tells us that the function $f(\theta)$ must be a constant value. Evaluating $f(0)$ gives,

$$f(0) = e^{0}(\cos 0 + \mathrm{j}\sin 0) = 1 \times 1 = 1,$$

$$e^{-\mathrm{j}\theta}(\cos\theta + \mathrm{j}\sin\theta) = 1,$$

$$e^{\mathrm{j}\theta} = \cos\theta + \mathrm{j}\sin\theta.$$

□

4 Phasors and Complex Amplitudes

We return to the idea of phasors and introduce an entity known as the complex amplitude. Phasors actually represent the following quantity,

where V is the RMS value, ω the angular frequency and φ the phase. Phasors are effectively just time-varying complex numbers. In particular, they rotate anticlockwise around the Argand diagram with positive time, which is illustrated in Figure 2. However, the rotation, whilst a good

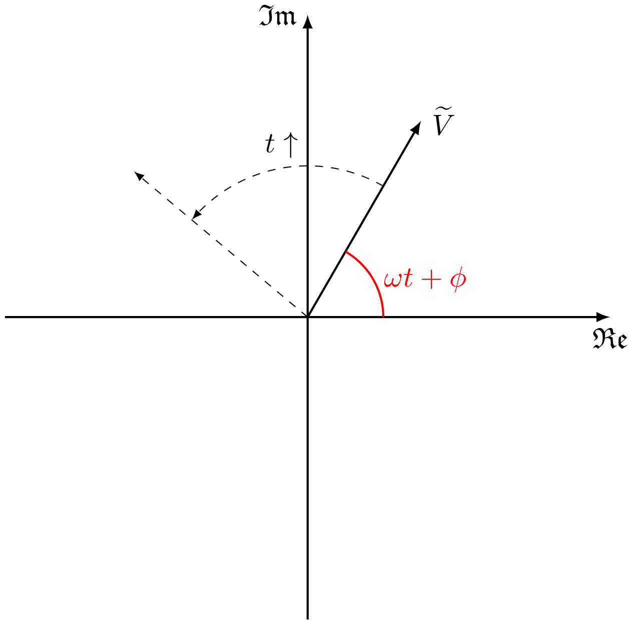


Figure 2: Illustration of a rotating phasor.

image, is slightly tedious to work with. We can manipulate the phasor definition into,

One can imagine that when solving an equation involving phasors, we can factor out the time-variant term (we will motivate this later), leaving only,

where V_c represents the complex amplitude. We will see that the complex amplitude is what is truly useful, as it gives information for both the magnitude of the signal, but also its phase shift, relative to some datum. Commonly, the datum will be defined such that the forcing function has zero phase shift.

Because phasors and complex amplitudes are intrinsically complex numbers, the rules of complex numbers apply. We will focus on complex addition. Mathematically speaking, the set of complex numbers, $\mathbb{C}$ is isomorphic to the set of 2D vectors (technically a planar vector space), $\mathbb{R}^2$. That is to say, there is symmetry between the two sets. In short, the claim is that vector addition and scalar multiplication behaves in identical fashion for 2D vectors and complex numbers, and indeed this is true,

This is much easier to imagine when comparing an Argand diagram with a set of Cartesian axes. The $\mathfrak{Re}$-axis quite literally lies in the same position as the x-axis, and similarly for $\mathfrak{Im}$ and y. Further links could be made between $[\widehat{\imath}, \widehat{\jmath}]$ and $[1, \mathrm{j}]$. This is building up to visualising complex number addition as shown in Figure 3. This idea will be useful later.

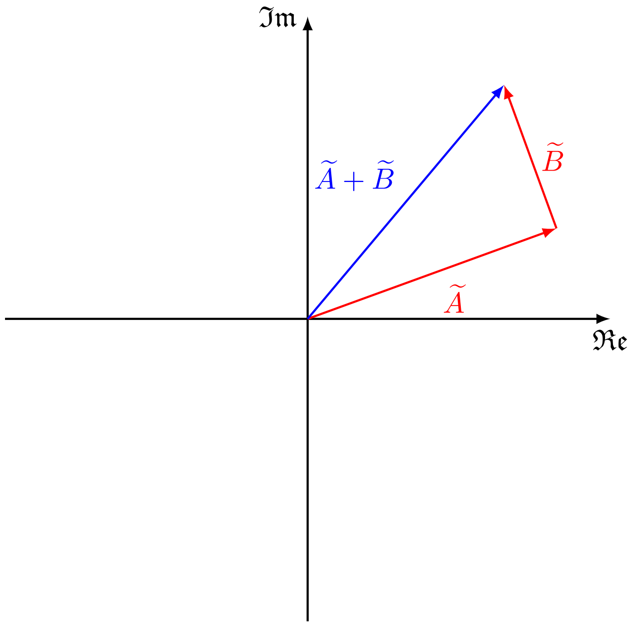


Figure 3: Phasor addition.

5 Period and Frequency

Let us return to the claim that the time variant term can be factored out. Half of the motivation will be performed here. In this section, we will prove that the response function will oscillate at the same frequency as the forcing function (almost always). Let us recall the general expression for a linear system,

We take $F_1(t)$ to be periodic with angular frequency ω. Let us reparameterise F_1 from $t \to \theta$, i.e.

$$F_1(t) = F_2[\theta(t)] = F_2(\theta),$$

where we define,

$$\theta = \omega t + \varphi.$$

F_2 is clearly still periodic, but now of period 2π. The general differential equation now becomes,

where β_k represents real constants. The symbol change denotes that α_k and β_k are not necessarily the same, but it does not matter. Since F is periodic, then $F_2(\theta) = F_2(\theta + 2n\pi)$. Let $\theta = \phi - 2n\pi$,

because $\frac{\mathrm{d}\phi}{\mathrm{d}\theta} = 1$. Now make a change of variables (relabelling) $\phi \to \theta$.

where $T = \frac{2\pi}{\omega}$. That is to say, the response function is also periodic with the same angular frequency as the forcing function. This means that in our current-voltage linear systems, every $e^{\mathrm{j}\omega t}$ will have the same ω.

5.1 Example: Forced Oscillations

Let us think about a mechanical system, since it is governed by the same type of linear differential equation.

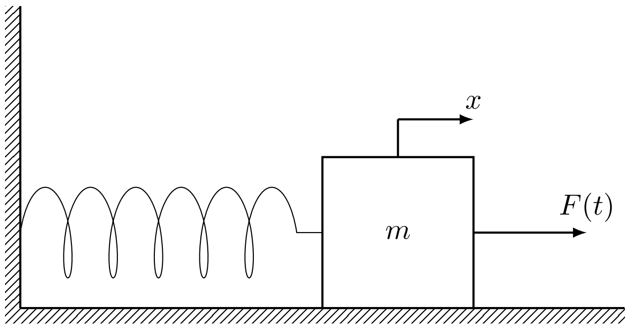


Figure 4: Forced mass-spring system.

Writing out the differential equation,

We know that in the case where $F(t) = 0$, this system must oscillate at $\omega_{\mathrm{N}} = \sqrt{\frac{k}{m}}$. Let us do the analysis for when $F(t) = \widehat{F}\cos(\omega t)$, Firstly, let us find the complementary function for when the DE is homogeneous. Since this is equivalent to the case where $F(t) = 0$,

Following this, we aim to find the particular integral, which directly depends on $F(t)$. As an anzatz, let $x_{\mathrm{PI}} = C\cos(\omega t)$.

We assume the response function to have a causal relationship with the forcing function. That is to say, if $F(t) = 0 \implies x = 0$.

6 Impedances

A priori, we know that the differential equations governing I-V characteristics for capacitors and inductors are given by,

Here v represents a potential drop, i.e. how much potential is consumed by the component. For a capacitor, a positive current will mean charging the capacitor, increasing the voltage across it, corresponding to a greater potential drop v. Hence, $q = Cv$ contains no negative sign. For an inductor, by Lenz's law, it generates a negative emf for positive current changes. However, a negative emf behaves like a positive potential drop, and so $v = L\frac{\mathrm{d}i}{\mathrm{d}t}$ contains no negative sign. Now, let us evaluate the derivatives using exponential functions, and

Let us recognise the similarity to Ohm's Law for resistors, $V \propto I$. For the original form of Ohm's Law, V and I were strictly real. However, we have now seen that V and I can take complex values during the process of calculation (as long as we take the real part at the end). We define the constant of proportionality Z, called impedance, for circuits with linear AC components (i.e. capacitors and inductors).

$$V = IZ.$$

Clearly, for a resistor, $Z = R$. For inductors and capacitors,

Since the standard "total resistance" formulae are defined purely by Ohm's Law and Kirchhoff's Laws (both of which are valid, Ohm's Law from above and Kirchhoff's Laws because they apply to the governing differential equations), they must also be valid as "total impedance" formulae, i.e.

$$Z_{\text{series}} = \sum Z_i,$$

$$\frac{1}{Z_{\text{parallel}}} = \sum \frac{1}{Z_i}.$$

Proof:

You can now see that, using complex impedance, the differential equations disappear and turn into simple linear equations with complex coefficients. This is the final part of the reason why we can cancel the $e^{j\omega t}$ terms from the equations, and why complex amplitudes are generally more useful than phasors.

6.1 Example: Calculating the Impedance of an RLC Circuit

To gain familiarity, let us calculate the impedance of the RLC circuit in Figure 5.

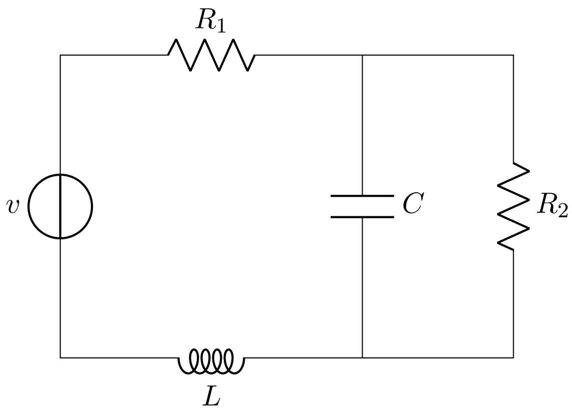


Figure 5: An RLC circuit.

6.2 Example: Calculating the Current of an RC Circuit

To illustrate the claim that AC circuit analysis is easier using complex exponentials than trigonometric functions, we shall calculate the current as a function of time for the simple RC circuit given in Figure 5. Firstly, let us do it with trigonometric functions only. Do not be concerned with the

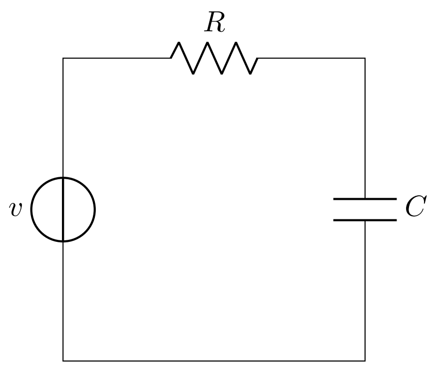


Figure 6: An RC circuit.

algebra, it is merely put here to truly emphasise the difficulty of this method.

$$v = \widehat{V}\cos(\omega t).$$

Kirchhoff's Loop Law:

$$\sum \varepsilon = \sum pd,$$

$$\widehat{V}\cos(\omega t) = iR + \frac{q}{C} = R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C},$$

$$\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{RC} = \frac{\widehat{V}}{R}\cos(\omega t).$$

We employ the integrating factor method of solving first order exact differential equations,

$$e^{\frac{t}{RC}}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{RC}e^{\frac{t}{RC}} = \frac{\widehat{V}}{R}e^{\frac{t}{RC}}\cos(\omega t),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(qe^{\frac{t}{RC}}\right) = \frac{\widehat{V}}{R}e^{\frac{t}{RC}}\cos(\omega t),$$

$$qe^{\frac{t}{RC}} = \int \frac{\widehat{V}}{R}e^{\frac{t}{RC}}\cos(\omega t)\mathrm{d}t,$$

$$q = \frac{\widehat{V}}{R}e^{-\frac{t}{RC}}\int e^{\frac{t}{RC}}\cos(\omega t)\mathrm{d}t.$$

Solving the integral using integration by parts. Let us define I_n,

$$I_n = \int e^{\frac{t}{RC}}\cos(\omega t)\mathrm{d}t,$$

$$I_n = e^{\frac{t}{RC}}\frac{\sin(\omega t)}{\omega} - \frac{1}{\omega RC}\int e^{\frac{t}{RC}}\sin(\omega t)\mathrm{d}t,$$

$$I_n = e^{\frac{t}{RC}}\frac{\sin(\omega t)}{\omega} - \frac{1}{\omega RC}\left[-e^{\frac{t}{RC}}\frac{\cos(\omega t)}{\omega} + \frac{1}{\omega RC}\int e^{\frac{t}{RC}}\cos(\omega t)\mathrm{d}t\right],$$

$$I_n = e^{\frac{t}{RC}}\frac{\sin(\omega t)}{\omega} - \frac{1}{\omega RC}\left[\frac{I_n}{\omega RC} - e^{\frac{t}{RC}}\frac{\cos(\omega t)}{\omega}\right],$$

$$\left(1 + \frac{1}{\omega^2R^2C^2}\right)I_n = \left(\frac{\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2RC}\right)e^{\frac{t}{RC}} + A,$$

where A is a constant of integration.

$$I_n = \left(\frac{\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2RC}\right)e^{\frac{t}{RC}}\left(1 + \frac{1}{\omega^2R^2C^2}\right)^{-1} + B,$$

$$q = \frac{\widehat{V}}{R}e^{-\frac{t}{RC}}I_n = \frac{\widehat{V}}{R}\left(\frac{\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2RC}\right)\left(1 + \frac{1}{\omega^2R^2C^2}\right)^{-1} + Be^{-\frac{t}{RC}}.$$

Steady AC analysis tells us $q(t) = q(t + \frac{2\pi}{\omega}) \Rightarrow B = 0$.

$$i = \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\widehat{V}}{R}\left(\sin(\omega t) + \frac{\cos(\omega t)}{\omega RC}\right)\left(1 + \frac{1}{\omega^2R^2C^2}\right)^{-1},$$

$$i = \frac{\widehat{V}}{R}\rho\cos(\omega t + \varphi)\left(1 + \frac{1}{\omega^2R^2C^2}\right)^{-1},$$

where

$$\rho^2 = 1 + \frac{1}{\omega^2R^2C^2},$$

$$\varphi = \arctan\left(\frac{1}{\omega RC}\right),$$

$$\therefore i = \frac{\widehat{V}}{R} \cdot \frac{\cos(\omega t + \varphi)}{\sqrt{1 + \frac{1}{\omega^2 R^2 C^2}}} = \frac{\widehat{V}\cos(\omega t + \varphi)}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}}.$$

That was a lot of work, let us redo it with complex exponentials.

$$v = \widehat{V} e^{\mathrm{j}\omega t},$$

$$Z_{\text{tot}} = R + \frac{1}{\mathrm{j}\omega C} = R - \frac{\mathrm{j}}{\omega C} = \sqrt{R^2 + \frac{1}{\omega^2 C^2}} e^{-\mathrm{j}\arctan\left(\frac{1}{\omega RC}\right)},$$

$$i = \frac{v}{Z_{\text{tot}}} = \frac{v}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} e^{\mathrm{j}\varphi},$$

$$\mathfrak{Re}\{i\} = \frac{\widehat{V}\cos(\omega t + \varphi)}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}}.$$

4 lines vs 20 lines, take your pick.

We can actual consider this problem differently using phasor diagrams and complex impedances. This is a minor adjustment, which just means that the problem is geometric and does not require converting complex numbers to polar form.

$$i = \frac{v}{Z_{\text{tot}}} = \frac{Rv + \frac{\mathrm{j}}{\omega C} v}{R^2 + \frac{1}{\omega^2 C^2}}.$$

Now factor out the $e^{\mathrm{j}\omega t}$,

$$I_{\text{c}} = \frac{RV_{\text{c}} + \frac{\mathrm{j}}{\omega C} V_{\text{c}}}{R^2 + \frac{1}{\omega^2 C^2}}.$$

We can draw this on a phasor diagram. We define V_c has zero phase shift, i.e. it is the datum. From the Figure 7, we can conclude,

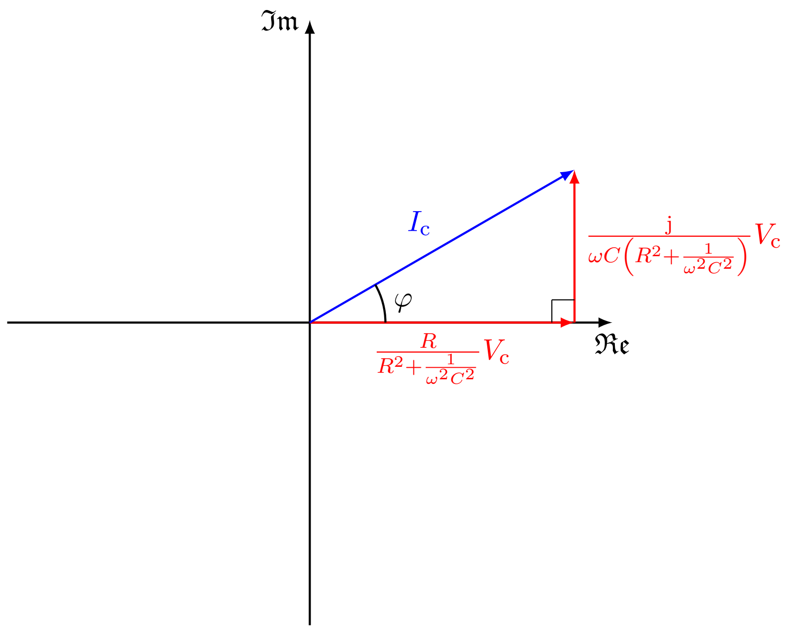


Figure 7: Phasor diagram for the RC circuit.

$$\varphi = \arctan\left(\frac{1}{\omega RC}\right),$$

$$|I_c| = \frac{V}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}},$$

where $V = V_c$ because V_c has a phase of 0. The right angle is because multiplying a number by j effectively rotates it anticlockwise by 90°. This is because $\mathrm{j} = e^{\mathrm{j}\frac{\pi}{2}}$.

6.3 Question: A Capacitor and Inductor in a Curious Link

This is a problem from the Jaan Kaalda notes [3].

Around a toroidal ferromagnetic core of a very large magnetic permeability, a coil is wound; this coil has a large number of loops and its total inductivity is L. A capacitor of capacitance C is connected to the middle point of the coil's wire as shown in figure. AC voltage V_0 of circular frequency ω is applied to the input leads of the circuit; what is the reading of the ammeter (which can be considered to be ideal)? This is shown in Figure 8.

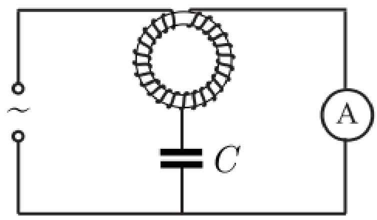


Figure 8: A toroidal inductor and a capacitor.

6.4 Question: Infinite LC Grid

This is a problem from the 1987 IPhO.

When sine waves propagate in an infinite LC grid (see the figure below) the phase of the AC voltage across two successive capacitors differs by φ. (a) Determine how φ depends on ω, L and C (ω is the angular frequency of the sine wave).

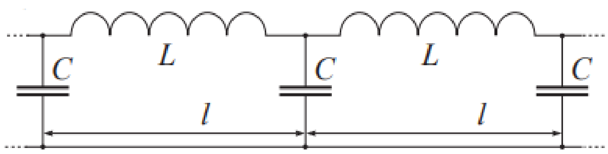


Figure 9: An infinite LC grid.

6.5 Question: Zero Impedance Frequency

This is a problem from the Jaan Kaalda notes [3].

Find such frequencies of the input voltage ω for which the circuits shown below have zero impedance.

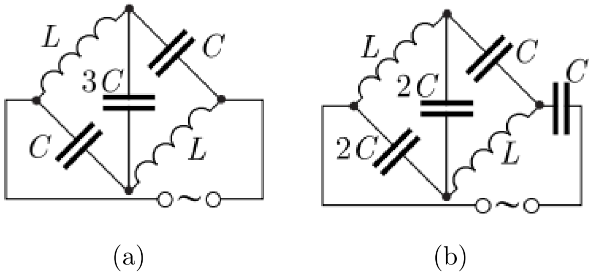


Figure 10: Circuits for 6.5.

6.6 Question: Maximal Amplitude

This is a problem from the Jaan Kaalda notes [3].

In the circuit shown in the figure, the sinusoidal input voltage has a fixed amplitude V_0 and frequency f. What is the maximal amplitude of the output voltage, and for which values of the variable resistances R_1, F_2, and R_3 is the maximal amplitude achieved?

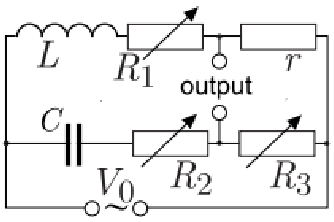


Figure 11: A variable impedance circuit.

6.7 Question: Black Box

This is a problem from the Jaan Kaalda notes [3].

In a black box with two ports, there are three components connected in series: a capacitor, an inductance, and a resistor. Devise a method to determine the values of all three components, if you have a sinusoidal voltage generator with adjustable output frequency ν, an AC-voltmeter and an AC-ammeter.

6.8 Question: Phase Shift

This is a problem from the Jaan Kaalda notes [3].

For the circuit shown below, the frequency of the sinusoidal input voltage is unknown; given the capacitance C, inductance L, resistances R_1, R_2, amplitude of the input voltage V_0, and the phase shift φ between the inductor current and input voltage, what is the phase shift between the capacitor voltage and output voltage?

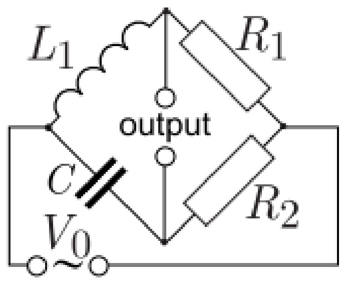


Figure 12: A bridge impedance circuit.

7 Filters

We have acquired expressions for the impedances of capacitors and inductors. Curiously, both of these contain a frequency term.

$$Z_{\mathrm{L}} = \mathrm{j}\omega L, \ Z_{\mathrm{C}} = \frac{1}{\mathrm{j}\omega C}.$$

For $\omega = 0$ rad s^{-1} (DC), we can see that $Z_{\mathrm{L}} = 0\ \Omega$ and $Z_{\mathrm{C}} \to \infty\ \Omega$. Intuitively, this makes sense, because we know that an inductor is just a coil of wire. In DC, the inductor behaves like a shorting wire, whereas, a capacitor is just an open circuit. When $\omega \to \infty$ rad s^{-1}, $Z_{\mathrm{L}} \to \infty\ \Omega$ and $Z_{\mathrm{C}} \to 0\ \Omega$. This is less intuitive, but we can still explain this using electromagnetism. The inductor resists change in magnetic flux, but since the change is infinite, the inductor generates an infinite counteracting emf. For a capacitor, the argument is a little more complex. These ideas give rise to the potential for frequency filters. At this point, it is sufficient to just know that they exist.

7.1 Example: A Simple RC Filter

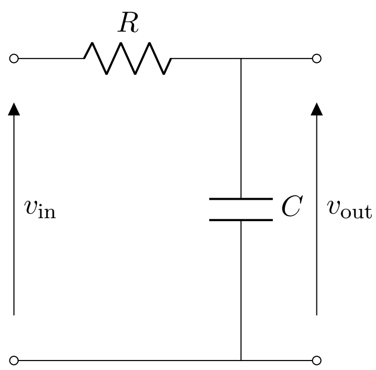


Figure 13: A first order low-pass filter.

Let us derive a relationship for the gain,

$$v_{\mathrm{out}} = v_{\mathrm{in}} \cdot \frac{Z_{\mathrm{C}}}{Z_{\mathrm{R}} + Z_{\mathrm{C}}},$$

$$G = \frac{v_{\mathrm{out}}}{v_{\mathrm{in}}} = \frac{1}{1 + \mathrm{j}\omega RC}.$$

The gain decreases with increasing frequency, and so allows low frequencies pass, i.e. it is a low-pass filter. We choose to define the cut-off frequency as the point where the loss in gain reaches 3 dB.

We define decibels as a logarithmic scale,

$$G_{\text{dB}} = 20 \log G \text{ dB}.$$

In this specific case,

$$\omega_{3\,\text{dB}} = \frac{1}{RC}.$$

7.2 Question: An Electronic Frequency Filter

This is a problem adapted from the 1984 IPhO.

An electronic frequency filter consists of four components as shown in figure: there are two capacitors of capacitance C, an inductor L, and a resistor R. An input voltage v_{in} is applied to the input leads, and the output voltage v_{out} is measured with an ideal voltmeter at the output leads, see figure. The frequency ν of the input voltage can be freely adjusted. Find the ratio of $v_{\text{out}}/v_{\text{in}}$ and the phase shift between the input and output voltages for the following cases: (a) at the limit of very high frequencies; (b) at the limit of very low frequencies.

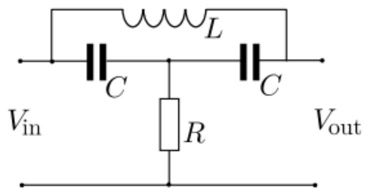


Figure 14: An electronic frequency filter.

8 Resonance

Let us return to the differential equation for a series RLC circuit.

$$L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{C} = v(t).$$

We can acquire the equation for an LC circuit by removing the R term, i.e.

$$L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + \frac{Q}{C} = v(t).$$

Assume $v = 0$ and we start the circuit by charging the capacitor or parse some residual current through the inductor.

You may recognise this as analogous to simple harmonic motion, where $\omega_{\text{N}}^2 = \frac{1}{LC}$. This is resonance. ω_{N} is the natural frequency, which is subtly different to the resonant frequency, ω_{res}. We will discuss this shortly. Now let us add the terms back, one at a time.

This corresponds to damped oscillations. Intuitively, it makes sense that R is the damping term, given that in the ideal case, R is the only component that dissipates heat. Should we solve this equation, we would discover that the damping term shifts the resonant frequency, lowering it. The natural frequency is defined as the resonant frequency of the system with no damping. Should you study vibrational mechanics further, there is an explicit relationship which can be derived between resonant frequency, natural frequency and the dampening ratio (a measure of damping). However, for IPhO, most questions thus far assume that the damping is low, so

$$\omega_{\text{res}} \approx \omega_{\text{N}} = \frac{1}{\sqrt{LC}}.$$

Let's reintroduce the final forcing term,

$$L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{C} = v(t).$$

As shown before, the circuit will always oscillate at the same frequency as the forcing term. Therefore, it is key to realise, the circuit will **not** resonate if $\omega_F \neq \omega_{\text{res}}$.

There are two types of resonance, namely current and voltage resonance. These are best understood with some example circuits, illustrated in Figures 15 and 16. Let us calculate the total impedances

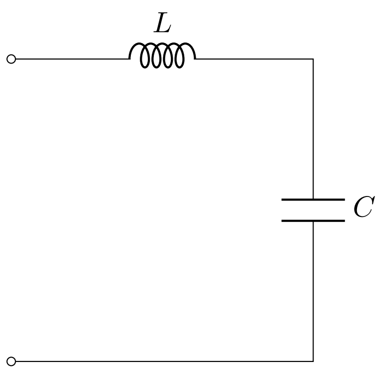


Figure 15: A series LC circuit.

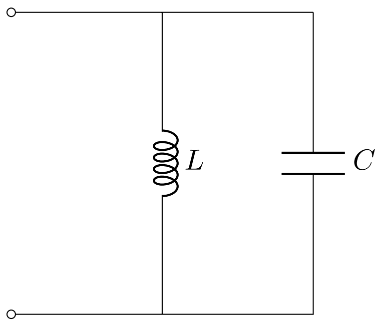


Figure 16: A parallel LC (shunt) circuit.

for both,

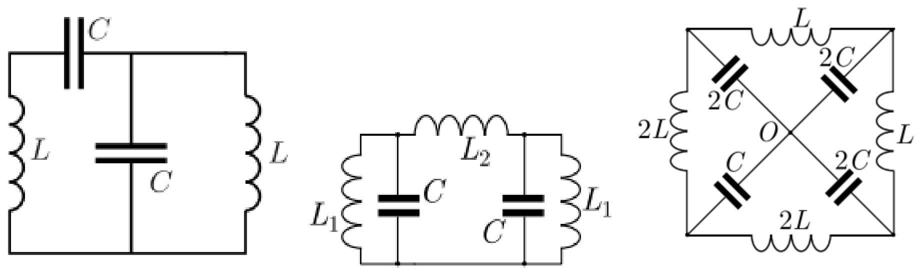
It is not hard to see that for $\omega = \frac{1}{\sqrt{LC}}$, $Z_\mathrm{s} = 0$ and $Z_\mathrm{p} \to \infty$. The easiest way of interpreting this is with Ohm's Law. In the series circuit, a small voltage will produce a large current (reciprocal of $\lim_{Z\to 0}$). On the other hand, in the parallel circuit, a small current will produce a large voltage (product with $\lim_{Z\to\infty}$). The former is current resonance, and the latter is voltage resonance, which are self-explanatory names.

In general, for any RLC circuit, resonance occurs when the total impedance becomes real, i.e. when the load becomes purely resistive. To calculate the resonance frequency of the given circuit, solve for when all the imaginary components cancel. Circuits can often have several modes of resonance. To find all of them, inspect all the linearly independent loops within the circuit.

So what is actually resonating? In electrical circuits, the resonating media are electromagnetic fields. The energy oscillates between electric and magnetic fields, as the charge is stored in the capacitors, and then dissipated as current through the inductors, which push charge back into the capacitors and etc.

8.1 Question: Finding the Natural Frequencies

This is a problem from the Jaan Kaalda notes [3].



(a) (b) (c)

Figure 17: Circuits for 8.1.

Find the natural frequencies of the circuit shown in the figures below.

8.2 Question: Finding the Natural Frequencies Pt.2

Determine all the natural frequencies of the circuit shown in Figure 18. You may assume that all the capacitors and inductances are ideal, and that the following strong inequalities are satisfied: $C_1 \ll C_2$, and $L_1 \ll L_2$. Note that your answers need to be simplified according to these strong inequalities.

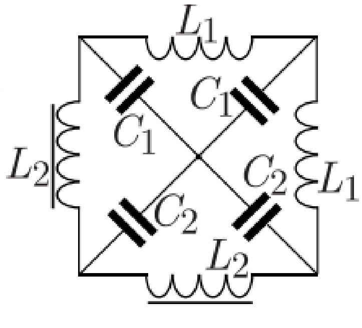


Figure 18: An LC circuit.

9 AC Power

Now, we will attempt to address the significance of RMS. To do this, we shall derive the average power for sinusoidal AC. Firstly, let us consider in-phase current and voltage, i.e. purely resistive loads. We define instantaneous power, p and average power, P. Time averages will be denoted with $\langle$angle braces$\rangle$.

$$p = iv = \widehat{I}\widehat{V}\cos^2(\omega t),$$

$$P = \langle p \rangle = \widehat{I}\widehat{V} \cdot \frac{1}{T}\int_0^T \cos^2(\omega t)\mathrm{dt},$$

$$P = \widehat{I}\widehat{V} \cdot \frac{\frac{1}{2}T}{T} = \frac{\widehat{I}\widehat{V}}{2}.$$

Since for **sinusoidal** signals, $X_{\mathrm{rms}} = \frac{\widehat{X}}{\sqrt{2}}$,

$$P = I_{\mathrm{rms}}V_{\mathrm{rms}} = IV.$$

This explains what is meant by "RMS values are those taken by a DC equivalent circuit", which is often quoted in school physics. This is unfortunately not yet the full picture. Now, we shall derive the average power for out-of-phase voltage and current. Without loss of generality, we assume voltage to have zero phase.

$$p = iv = \widehat{I}\widehat{V}\cos(\omega t)\cos(\omega t + \varphi),$$

$$P = \langle p \rangle = \widehat{I}\widehat{V} \cdot \frac{1}{T}\int_0^T \cos(\omega t)\cos(\omega t + \varphi)\mathrm{dt}.$$

We can employ a trigonometric identity,

$$\cos(A)\cos(B) = \frac{\cos(A+B) + \cos(A-B)}{2},$$

$$\Rightarrow P = \widehat{I}\widehat{V} \cdot \frac{1}{2T}\int_0^T \left(\cos(2\omega t + \varphi) + \cos(\varphi)\right)\mathrm{dt}.$$

Because the first cosine term is periodic with 2ω, it will time average to 0, and hence we are left with,

$$P = \frac{\widehat{I}\widehat{V}}{2} \cdot \cos\varphi = IV\cos\varphi.$$

In general, φ is the phase difference between the voltage and current. We call $\cos\varphi$ the power factor, and in general this should be kept as close to unity as possible. This is simply because a higher power factor means that VI can be less yet still supply the same power. The importance lies in minimising I, as (according to $P = I^2R$) currents cause large dissipation in transmission.

10 Further Reading

There are many books and papers which cover the described materials. Here are a selection of notes, which I believe are exceptionally good, and recommend reading following my own.

- Jaan Kaalda's notes on electricity [3] (and in general) are widely regarded as the go to material for any student training for IPhO. Jaan Kaalda is the Estonian IPhO team leader, and was previously the Head of Theoretical Examination for the IPhO.
- Kevin Zhou's notes on electromagnetism [6] are very content heavy, but provide excellent coverage and questions. Kevin Zhou is the American IPhO team leader, a team that frequently achieves Gold medals.
- Antonin Machacek's book "Upgrade Your Physics" [4] provides a brief but accurate summary of the topic, while covering many other areas. Antonin Machacek is a professor of physics at the University of Cambridge. In 1993, he came 10th in the IPhO.
- John P. Cullerne and Antonin Machacek's book "The Language of Physics" [2] is also another excellent resource for general physics study.
- "Fundamentals of Physics" [1] contains many elementary problems and good concise descriptions of IPhO syllabus content.
- Last but not least, Richard Feynmann's Lectures [5] are without a doubt truly some of the greatest material ever fabricated for physics. Volume II contains fundamentals for AC theory.

References

[1] Robert Resnick Jearl Walker, David Halliday. *Fundamentals of Physics*. Wiley, 2013.

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[3] Jaan Kalda. Electrical circuits. `https://www.ioc.ee/~kalda/ipho/electricity-circuits.pdf`, 2017. Accessed: 31-03-2024.

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[6] Kevin Zhou. Electromagnetism vi: Circuits. `https://knzhou.github.io/handouts/E6.pdf`. Accessed: 31-03-2024.